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Approximate Selections for Upper Semicontinuous Convex Valued Multifunctions

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1. INTRODUCTION

Let $\langle X, d_1 \rangle$ and $\langle Y, d_2 \rangle$ be metric spaces, and $\mathrm{CL}(Y)$ denote the closed nonempty subsets of Y. By a *multifunction* from X to Y we mean a function $\Gamma: X \to \mathrm{CL}(Y)$. By a *selection* f for Γ we mean a function $f: X \to Y$ such that for each x, $f(x) \in \Gamma(x)$. The systematic study of continuous selections begins with the papers of Michael (see, e.g., [12]); a survey of the literature on measurable selections (with respect to some σ-algebra of subsets on X) has been compiled by Wagner in [17] and [18].

The term *approximate selection* means different things to different people. Relative to the work of Michael [12], Deutsch and Kenderov [7], and Olech [14], an approximate selection for Γ is a function $f: X \to Y$ such that at each x in X, $f(x)$ is close to some point of $\Gamma(x)$. We are interested in a rather different notion studied by Cellina [4–6] and Reich [15], where an approximate selection f for Γ is one such that the graphs of f and Γ are "close," where close is defined in a strong or weak sense. Explicitly, if C is a set in a metric space, let $S_\varepsilon[C]$ denote the union of all open ε-balls whose centers run over C. Metrize $X \times Y$ using the metric ρ defined by $\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$. Identifying $f: X \to Y$ and $\Gamma: X \to \mathrm{CL}(Y)$ with their graphs, we say f *weakly ε-approximates* Γ if $S_\varepsilon[\Gamma] \supset f$. If, in addition, we have $S_\varepsilon[f] \supset \Gamma$, then we say that f *strongly ε-approximates* Γ.

The results of Cellina and Reich are restricted to a particular class of multifunctions. We call $\Gamma: X \to Y$ *upper semicontinuous* (u.s.c.) at x in X if for each $\varepsilon > 0$ there exists $\lambda > 0$ such that $\Gamma(S_\lambda[x]) \subset S_\varepsilon[\Gamma(x)]$. (A stronger requirement would be that given any neighborhood V of $\Gamma(x)$ there exists $\lambda > 0$ such that $\Gamma(S_\lambda[x]) \subset V$. In the literature this property is usually called upper semicontinuity, whereas ours is called Hausdorff upper semicontinuity [8].) Let Y be a normed linear space. Basically, Reich and Cellina have

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asked: When does there exist either a strong or weak ε-approximate selection for an u.s.c. convex valued multifunction $\Gamma: X \to \mathrm{CL}(Y)$? The existence of continuous strong ε-approximate selections for such multifunctions is also considered in the present paper, and we obtain a much more inclusive result than the main theorem of [5]. We also ask a different question: If a continuous strong ε-approximate selection for Γ does not exist, can we still strongly ε-approximate Γ by relatively nice functions? In the sequel the term ε-approximate selection shall mean a strong ε-approximate selection in the above sense.

Before proceeding we set forth some additional notation and terminology. Let X and Y be metric spaces and let $f: X \to Y$ be arbitrary. Denote the closure of f as a subset of $X \times Y$ by $\bar{f}$. For each x in X the *limit set* $L(f, x)$ of f at x is $\{y: (x, y) \in \bar{f}\}$. By the sequential characterization of the closure of a set in a metric space

$$L(f, x) = \{y: \exists \{x_n\} \to x \text{ for which } \{f(x_n)\} \to y\}.$$

Using the trivial sequence $x_1 = x$, $x_2 = x, \ldots$, we see that $f(x) \in L(f, x)$. If f is continuous at x, then $L(f, x) = \{f(x)\}$, but not conversely. The class of functions f for which $L(f, x) = \{f(x)\}$ for all x is simply those functions with closed graph; these are the subject of a recent monograph of Hamlett and Herrington [10]. The multifunction $\Gamma_f: X \to \mathrm{CL}(Y)$ defined by $\Gamma_f(x) = L(f, x)$ will be called the *limit set multifunction* induced by f. If $\Gamma: X \to \mathrm{CL}(Y)$ and $\Gamma = \Gamma_f$ for some f, we call f a *dense selection* for Γ [2]. If $f: X \to Y$ we denote the set of points of discontinuity of f by $D(f)$. The function f is said to be of *Baire class one* if the inverse image of each open subset of Y is an F_σ subset of X. Since each open subset of X is an F_σ set, the Baire class one functions include the continuous ones. For a thorough study of this class of functions, the reader should consult [11] (where these functions are called B-measurable of class one). If $f: X \to R$, the *support* of f, denoted by $\mathrm{supp}(f)$, is the closure of the set $\{x: f(x) \neq 0\}$.

The closure, set of limit points, and interior of a set C in a metric space will be denoted by $\bar{C}$, C', and int C, respectively. If K is another set in the metric space and there exists $\varepsilon > 0$ for which both $S_\varepsilon[C] \supset K$ and $S_\varepsilon[K] \supset C$, then the *Hausdorff distance* δ between C and K is given by

$$\delta[C, K] = \inf\{\varepsilon: S_\varepsilon[C] \supset K \text{ and } S_\varepsilon[K] \supset C\}.$$

Further information on this notion of distance can be found in Berge [3], Kuratowski [11], and Nadler [13].

Once again let Y be a normed linear space. Even when X and Y are very nice we cannot ε-approximate each u.s.c. convex valued multifunction

$\Gamma: X \to \mathrm{CL}(Y)$ by continuous functions. For example, $\Gamma: [0, 1] \to \mathrm{CL}(\mathrm{R})$ defined by

$$\Gamma(x) = R, \quad \text{if} \quad x = 0,$$
$$= \{0\}, \quad \text{otherwise},$$

admits no such approximations. However, for each $\varepsilon > 0$, Γ as described above can be ε-approximated by a discontinuous function. Ignoring Borel classification issues for the moment, there are certain obvious necessary conditions that a convex valued u.s.c. multifunction $\Gamma: X \to \mathrm{CL}(Y)$ must meet to admit some ε-approximate selection. Fix x in X. By Zorn's lemma there is a subset W of $\Gamma(x)$ such that for each y_1 and y_2 in W, $\|y_1 - y_2\| \geqslant 2\varepsilon$ and $\Gamma(x) \subset S_{2\varepsilon}[W]$. Now if $\{x\} \times \Gamma(x)$ is to be a subset of $S_\varepsilon[f]$ for some $f: X \to Y$, it follows that the cardinality of $S_\varepsilon[x]$ must be at least that of W. In particular, Γ must map isolated points to singletons, limit points that are not condensation points to separable sets, and so forth. In order to state "nontechnical" results valid for multifunctions defined on an arbitrary metric space X, we choose to require that the values of Γ be separable subsets of Y. Thus, our cardinality conditions reduce to the single condition: Γ maps isolated points to singletons.

We will show that if X is a metric space, Y is a normed linear space, and $\Gamma: X \to \mathrm{CL}(Y)$ is an u.s.c. multifunction mapping isolated points to singletons such that for each x, $\Gamma(x)$ is a separable convex set, then Γ can be ε-approximated by a Baire class one function whose limit set multifunction is both u.s.c. and convex valued. Put differently, the convex valued u.s.c. multifunctions that admit Baire class one dense selections are dense in the separable convex valued u.s.c. multifunctions, equipped with the Hausdorff metric topology as applied to their graphs. Moreover, if Γ has totally bounded values, then for each $\varepsilon > 0$ there exists a continuous $f: X \to Y$ such that $\delta[f, \Gamma] < \varepsilon$. If $Y = R^n$, we will show that for each $\varepsilon > 0$ there exists a Baire class one ε-approximate selection for Γ with a closed graph.

2. Preliminary Lemmas

A prime use of locally finite covers and partitions of unity subordinated to these covers [9, p. 170] is to piece together continuous functions defined locally to obtain a globally continuous function with prescribed properties. Specifically, let $\{U_i: i \in I\}$ be such a cover of X, let $\{p_i(\cdot): i \in I\}$ be a partition of unity subordinated to the cover, and for each i let $f_i: U_i \to Y$ (where Y is a normed linear space) be continuous. For each i we understand the symbol $p_i f_i$ to represent a function on X (rather than on just U_i) by requiring that $p_i f_i(x)$ be zero off U_i. Then $f\colon X \to Y$ defined by

$$f(x) = \sum_{i \in I} p_i f_i$$

is well defined and continuous. Our first two lemmas show that if we piece together discontinuous functions defined locally, then certain common qualitative aspects of their limit set structure are often preserved.

LEMMA 1. *Let X be a metric space and let Y be a normed linear space. Let Ω be a family of closed subsets of Y closed under translations and maps of the form $y \to \alpha y$ ($\alpha \geqslant 0$). Let $\{U_i : i \in I\}$ be a locally finite open cover of X, and let $\{p_i(\cdot) \colon i \in I\}$ be a partition of unity subordinated to the cover. Suppose for each index i, $f_i \colon U_i \to Y$ has the following properties*:

(1) *For each x in U_i, $L(f_i, x) \in \Omega$.*

(2) *For each x in X, at most one f_i is discontinuous at x.*

(3) *For each x in U_i, $p_i(x) = 0$ implies f_i is continuous at x.*

Suppose $f = \sum p_i f_i$. Then for each x in X we have $L(f, x) \in \Omega$.

Proof. Since locally there exist indices $i_1, i_2, \ldots, i_n$ such that $f = \sum_{j=1}^{n} p_{i_j} f_{i_j}$ and for each open set V the limit sets of $f \mid V$ agree with the limit sets of f at each point of V, it suffices to show that for each i and k in I the function $g = p_i f_i + p_k f_k$ has limit sets in Ω. We first show this to be true for $p_i f_i$. We consider the possible locations of a variable point x on X. If $x \notin \operatorname{supp}(p_i)$, then $p_i f_i$ is zero in a neighborhood of x, whence $L(p_i f_i, x)$ is a singleton, and therefore in Ω. If $p_i(x) \neq 0$, then $x \in U_i$ and $L(p_i f_i, x) = p_i(x) L(f_i, x)$, a homothetic image of $L(f_i, x)$. Here, too, $L(p_i f_i, x)$ is in Ω. Finally, if $x \in \operatorname{supp}(p_i)$ and $p_i(x) = 0$, then by condition (3) $L(p_i f_i, x) = \{0\}$. We now show $L(g, x) \in \Omega$ at each x. This is clearly true if g is continuous at x. Otherwise, w.l.o.g., we may assume $p_i f_i$ is discontinuous at x. Then $x \in U_i$ and f_i is discontinuous at x. By (2), f_k and therefore $p_k f_k$ are continuous at x. It now follows that $L(g, x) = L(p_i f_i, x) + p_k f_k(x)$, a set which is again in Ω by the first part of the proof.

There are many possibilities for the family Ω of Lemma 1: the singletons, the convex sets, the star-shaped sets, the bounded sets, the finite sets, the flats, etc. Of course, we will be interested in the first two configurations just listed. To appreciate the need for conditions (2) and (3) in the statement of Lemma 1, we present two simple constructions.

EXAMPLE 1. Let $f\colon R \to R$ be defined by

$$f(x) = \frac{1}{x}, \qquad \text{if} \quad x \neq 0,$$
$$= 0, \qquad x = 0.$$

Then f has singleton limit sets, i.e., its graph is closed. However, if $p(x) = x$, then $L(pf, 0) = \{0, 1\}$, a nonconvex set. If $h\colon R \to R$ is defined by $h(x) = -f(x)$ if $x \neq 0$ and $h(0) = 1$, then h also has singleton limit sets whereas $L(f + h, 0) = \{0, 1\}$.

LEMMA 2. *Let X be a metric space and let Y be a normed linear space. Let $\{U_i : i \in I\}$ be a locally finite open cover of X, and let $\{p_i(\cdot) : i \in I\}$ be a partition of unity subordinated to the cover. Suppose for each $i \in I$, $f_i\colon U_i \to Y$ has the following properties:*

(1) *The limit set mulfunction Γ_i for f_i is* u.s.c. *on U_i.*

(2) *For each pair of distinct indices i and k, whenever $x \in D(f_i)$, then $x \notin$* supp(p_k).

Then the limit set multifunction for $f = \sum p_i f_i$ is u.s.c. *on X.*

Proof. As in the proof of Lemma 1 it suffices to show that for each i and k the limit set multifunction for $g = p_i f_i + p_k f_k$ is u.s.c. First, we show that the limit set multifunction for $p_i f_i$ is u.s.c. If $p_i f_i$ is continuous at x, we obviously have upper semicontinuity at x. Otherwise, by condition (2) and the definition of partition of unity, the multifunction agrees locally with Γ_i and must be also u.s.c. by (1). To show the limit set multifunction for g is u.s.c., fix x in X and let $\varepsilon > 0$. W.l.o.g. we may assume that $p_k f_k$ is continuous at x. By the first part of the proof there exists $\lambda > 0$ such that whenever $d(x, z) < \lambda$, then $L(p_i f_i, z) \subset S_{\varepsilon/2}[L(p_i f_i, x)]$ and $\|p_k f_k(z) - p_k f_k(x)\| < \varepsilon/2$. From the above inclusion, whenever $d(z, x) < \lambda$, then $p_i f_i(z) \in S_{\varepsilon/2}[L(p_i f_i, x)]$, and it follows that

$$\begin{aligned} g(z) &\in S_{\varepsilon/2}[L(p_i f_i, x)] + S_{\varepsilon/2}[p_k f_k(x)] \\ &\subset S_\varepsilon[L(p_i f_i, x) + p_k f_k(x)] = S_\varepsilon[L(g, x)]. \end{aligned}$$

This implies that $L(g, z) \subset S_\varepsilon[L(g, x)]$ whenever $d(z, x) < \lambda$.

Although a somewhat weaker condition may be substituted for condition (2) of Lemma 2, conditions (2) and (3) of Lemma 1 do not suffice.

EXAMPLE 2. Let $Y = l_2$, the Hilbert space of square summable real sequences, with the usual norm. Let C be the following closed convex subset of Y: $C = \{\{a_i\}$: for each $i \in Z^+$, $a_i \leqslant i\}$. Define $\Gamma: R \to \mathrm{CL}(Y)$ to be the constant multifunction $\Gamma(x) \equiv C$. By Theorem 5 of [2] Γ has a dense (Baire class two) selection f, i.e., for each x in R, $L(f, x) = C$. It is easy to check that for each $\alpha > 1$ and each $\varepsilon > 0$, the set $S_\varepsilon[C]$ fails to contain αC. Hence if $p: R \to (0, 1)$ is an arbitrary strictly increasing function, it follows that $x \to L(pf, x) = p(x)C$ fails to be (right) u.s.c. at any point of R. Hence, although $p(x)$ is positive at each point of discontinuity of f, pf fails to have an u.s.c. limit multifunction.

Our next two lemmas involve the local definition of functions.

LEMMA 3. *Let X be a metric space and let Y be a normed linear space. Let $x_0 \in X'$ and let K be a closed ball with center x_0. If $V \supset K$ is open and $C \subset Y$ is a separable closed convex set, then for each $\varepsilon > 0$ there exists $h: V \to C$ such that $\delta[h(K), C] \leqslant \varepsilon$ and*

(1) $D(h) = \{x_0\}$.

(2) *The limit set multifunction for h is* u.s.c. *and convex valued.*

If C is totally bounded, then h can be chosen continuous.

Proof. Suppose first that C is totally bounded. Choose $\{y_1, y_2, \ldots, y_n\}$ in C such that $S_\varepsilon[\{y_1, y_2, \ldots, y_n\}] \supset C$. Let $\{x_1, x_2, \ldots, x_n\} \subset K$ be arbitrary. By the Dugundji extension theorem [9, p. 188] there exists a continuous $h: V \to C$ such that for $j = 1, 2, \ldots, n$, $h(x_j) = y_j$. Clearly, $\delta[h(K), C] \leqslant \varepsilon$. If C is not totally bounded, let $\{y_j: j \in Z^+\}$ be a countable dense subset of C. Let $\{x_j\}$ be a sequence in K convergent to x_0, and let $\{S_{\lambda_j}[x_j]: j \in Z^+\}$ be pairwise disjoint open balls none of which contains x_0. For each $k \in Z^+$ let $E_k = \{p: p \in Z^+$ and $p = 2^{k-1}q$, where 2 and q are relatively prime$\}$. For each $j \in Z^+$ define $h(x_j)$ to be y_k, where k is the unique integer for which $j \in E_k$. Now extend h to V as follows:

$$h(x) = y_1 + \left[1 - \frac{1}{\lambda_j} d(x, x_j)\right](h(x_j) - y_1), \qquad \text{if} \quad d(x, x_j) < \lambda_j,$$

$$= y_1, \qquad \text{otherwise.}$$

Notice that for each x in V, $h(x)$ is a convex combination of y_1 and y_k for some $k > 1$. Thus, $h(V) \subset C$. Since $\lim_{j \to \infty} \lambda_j = 0$ and $\lim_{j \to \infty} x_j = x_0$, it is evident that x_0 is the only point of discontinuity of h. By the construction for each $k \in Z^+$ the point y_k in is $L(h, x_0)$, and since $L(h, x_0) \subset \overline{h(V)} \subset C$, it follows that $L(h, x_0) = C$. The upper semicontinuity of $x \to L(h, x)$ on V is obvious, as is $\delta[h(K), C] = 0$.

LEMMA 4. *Let X be a metric space and let R^n be n-dimensional*

Euclidean space. Let $x_0 \in X'$ and let K be a closed ball with center x_0. If $V \supset K$ is open and $C \subset R^n$ is a closed convex set, then there exists $h: V \to C$ with a closed graph such that $\delta[h(K), C] \leqslant \varepsilon$.

Proof. W.l.o.g. we can assume that $0 \in C$. If C is bounded, then C is totally bounded, and we are done by Lemma 3. Otherwise, by Theorem 8.4 of [16] there is a nonzero vector y_0 in C such that for each y in C, $y + y_0$ is again in C (such a vector is called a *direction of recession* for C). Let $\{y_j : j \in Z^+\}$ be a countable subset of C such that $C \subset S_\varepsilon[\{y_j : j \in Z^+\}]$, and whenever $k \neq j$, $\|y_k - y_j\| \geqslant \varepsilon$. Since this set is closed and discrete, for each $n \in Z^+$ only finitely many elements can lie in $S_n[0]$. Let $\{x_j\}$ and $\{S_{\lambda_j}[x_j] : j \in Z^+\}$ be defined as in the proof of Lemma 3. For each $j \in Z^+$ define $h_j : \{x : 0 < d(x, x_j) < \lambda_j\} \to C$ by

$$h_j(x) = jy_0 + \frac{y_0}{d(x, x_j)[\lambda_j - d(x, x_j)]}.$$

Notice that as x approaches either x_j or the boundary of $S_{\lambda_j}[x_j]$, $\|h_j(x)\|$ approaches infinity. Finally, define $h: V \to R^n$ by

$$\begin{aligned} h(x) &= y_j, && \text{if for some } j,\ x = x_j, \\ &= h_j(x), && \text{if for some } j,\ 0 < d(x, x_j) < \lambda_j, \\ &= y_0, && \text{otherwise.} \end{aligned}$$

By the above remarks concerning each h_j, the closedness of the graph of h is only at issue at $x = x_0$. However, since $\inf\{\|h_j(x)\| : 0 < d(x, x_j) < \lambda_j\} > j\|y_0\|$ and $\lim_{j \to \infty} \|h(x_j)\| = \infty$, whenever $\{z_k\} \to x_0$, then either $\{h(z_k)\} \to y_0$ or $\{h(z_k)\}$ fails to converge. Thus, $L(h, x_0) = \{y_0\} = \{h(x_0)\}$, and the graph of h is closed. Clearly, $h(V) \subset C$, and since $h(\{x_j : j \in Z^+\}) = \{y_j : j \in Z^+\}$, we have $\delta[h(K), C] \leqslant \varepsilon$.

We need one more lemma before our main results. It is a key one.

LEMMA 5. *Let X be a metric space with $(X')'$ nonempty. Let V be an open set containing $(X')'$. Then there exists a pair of open sets G_1 and G_2 such that $G_1 \cap G_2 = \varnothing$, $G_1 \cup G_2 = X$, and $(X')' \subset G_1 \subset V$.*

Proof. Since $(X')'$ is closed and $X - V$ is closed, by normality there exist disjoint open sets U and U^* such that $X - V \subset U$ and $U^* \supset (X')'$. For each $x \in X' - V$ there exists $\varepsilon_x \in (0, 1)$ such that $S_{\varepsilon_x}[x]$ contains no other limit point of X and $S_{\varepsilon_x}[x] \subset U$. For each such x let $\lambda_x = \frac{1}{2}\varepsilon_x$. We now show that the open set

$$H = \bigcup \{S_{\lambda_x}[x] : x \in X' - V\}$$

is also closed. Suppose $\{z_n\}$ is a sequence in H convergent to some point z. For each n choose $x_n \in X' - V$ such that $d(x_n, z_n) < \lambda_{x_n}$. Now $\lim_{n\to\infty} d(x_n, z) \neq 0$, or otherwise $z \in (X')'$. However, for each n, $z_n \in U$, whence $z \in X - U^*$. This contradicts $(X')' \subset U^*$. By passing to a subsequence we can assume $\lim_{n\to\infty} d(x_n, z)$ exists and is positive: We then have

$$\begin{aligned} 0 < \lim_{n\to\infty} d(x_n, z) &= \lim_{n\to\infty} d(x_n, z_n) \\ &\leqslant \liminf_{n\to\infty} \lambda_{x_n} \\ &< \liminf_{n\to\infty} \varepsilon_{x_n}. \end{aligned}$$

In particular, there exists $N \in Z^+$ such that $d(x_N, z) < \varepsilon_{x_N}$. Thus, unless $\{z_n\}$ is constant eventually, $S_{\varepsilon_{x_N}}[x_N]$ contains a limit point of X in contradiction to the choice of ε_{x_N}. Thus $\{z_n\}$ must be constant eventually, and $z \in H$ follows.

Finally, let $G_1 = V - H$ and let $G_2 = X - G_1$. From the preceding discussion, G_1 is open. The set G_2 consists of the open set H plus the isolated points of X belonging to neither G_1 nor H. This latter set is clearly open; so, G_2 is open.

3. The Main Results

The vehicle we use to pass from functions $\{f_i : i \in I\}$ defined on elements of some locally finite open cover $\{U_i : i \in I\}$ of X that are each close locally to a convex valued u.s.c. multifunction Γ to obtain an ε-approximate selection f for Γ defined globally is a slight modification of an argument buried in the proof of the main theorem of [5]. We single it out as a lemma.

Cellina's Lemma. *Let X be a metric space and let Y be a normed lionear space. Let $\Gamma: X \to \mathrm{CL}(Y)$ be* u.s.c. *and convex valued. Suppose there exists a locally finite open cover $\{U_i : i \in I\}$ of X, and for each $i \in I$ a closed set K_i, a point b_i, a number λ_i, and a function $f_i: U_i \to \Gamma(b_i)$ such that*

(1) $K_i \subset U_i \subset S_{\lambda_i/2}[b_i]$.

(2) $\lambda_i < \varepsilon$ *and* $\Gamma(S_{\lambda_i}[b_i]) \subset S_{\varepsilon/2}[\Gamma(b_i)]$.

(3) *Whenever* $i \neq k$, *then* $K_i \cap U_k = \varnothing$.

(4) $\delta[f_i(K_i), \Gamma(b_i)] \leqslant \varepsilon/2$.

Suppose $\{p_i(\cdot): i \in I\}$ is a partition of unity subordinated to the cover $\{U_i : i \in I\}$. Then the function $f = \sum p_i f_i$ satisfies $\delta[f, \Gamma] \leqslant \varepsilon$.

In the proof of the main theorem of [5], X is a metric locally convex space, each set K_i is convex, each f_i is continuous, and Γ is assumed to have totally bounded values. However, none of these assumptions are used to prove the above lemma (the details of which are left to the reader), a fact we shall now exploit in conjunction with the results of the last section.

THEOREM 1. *Let X be a metric space and let Y be a normed linear space. Let $\Gamma: X \to \mathrm{CL}(Y)$ be an* u.s.c. *multifunction with the following properties*:

(1) *Γ maps isolated points of X to singletons.*

(2) *For each x in X, $\Gamma(x)$ is a separable convex set.*

Then there exists a Baire class one function $f: X \to Y$ whose limit set multifunction is u.s.c. *and convex valued such that $\delta[f, \Gamma] \leq \varepsilon$. If the values of Γ are each totally bounded, then f can be chosen continuous.*

Proof. We first consider metric spaces X for which $(X')'$ is empty. If X' is empty, then by (1) Γ itself is a continuous single valued function, and there is nothing to prove. Otherwise, for each x in X' choose $\lambda_x > 0$ such that (i) $S_{\lambda_x}[x]$ contains no other limit point of X, (ii) $\lambda_x < \varepsilon$, and (iii) if $d(x, z) < \lambda_x$, then $\Gamma(z) \subset S_{\varepsilon/2}[\Gamma(x)]$. Set $H = \bigcup\{S_{\lambda_x}[x]: x \in X'\}$. Since each point of H belongs to a unique ball $S_{\lambda_x}[x]$, the open cover $\{S_{\lambda_x}[x]: x \in X'\}$ of H is locally finite. The only partition of unity $\{p_x(\cdot): x \in X'\}$ subordinated to the cover is the one for which each function p_x is simply the characteristic function of $S_{\lambda_x}[x]$. For each x in X' let $K_x = \{z: d(z, x) \leq \frac{1}{2}\lambda_x\}$. By Lemma 3 there exists $f_x: S_{\lambda_x}[x] \to \Gamma(x)$ such that (i) $\delta[f_x(K_x), \Gamma(x)] \leq \varepsilon/2$, (ii) $D(f_x) = \{x\}$, and (iii) the limit set multifunction for f_x is convex valued and u.s.c. If the values of Γ are totally bounded, then each such function can be chosen continuous. Let $f_1: H \to Y$ be given by $f_1 = \sum p_x f_x$. Note that for each $z \in H$, $f_1(z) = f_x(z)$, where x is the unique limit point of X for which $d(z, x) < \lambda_x$. By Cellina's lemma $\delta[f_1, \Gamma | H] \leq \varepsilon$. The hypotheses of Lemmas 1 and 2 are also satisfied; so, the limit set multifunction for f_1 is both u.s.c. and convex valued. Also, since $D(f_1)$ is a closed discrete set, f_1 is of Baire class one, and if each f_x is continuous, so is f_1.

The points of X not in H are isolated; so, for each such point x the set $\Gamma(x)$ is a singleton. Define $f_2: X - H \to Y$ by $f_2(x) = \Gamma(x)$. Finally, define $f: X \to Y$ by

$$
\begin{aligned}
f(x) &= f_1(x), &&\text{if} \quad x \in H, \\
&= f_2(x), &&\text{if} \quad x \in X - H.
\end{aligned}
$$

Since $D(f) = D(f_1)$, the function f is of Baire class one. If f_1 is continuous, then since H and $X - H$ form a separation of X, the function f will be continuous, too. For the same reason, the values of f_1 (resp. f_2) have no bearing on the limit sets of f_2 (resp. f_1), whence the limit set multifunction for f is both u.s.c. and convex valued. Clearly, $\delta[f, \Gamma] \leqslant \varepsilon$.

We now consider X for which $(X')'$ is nonempty. For notational simplicity we write F for $(X')'$. For each x in F choose $\lambda_x < \varepsilon$ such that if $z \in X$ and $d(z, x) < \lambda_x$, then $\Gamma(z) \subset S_{\varepsilon/2}[\Gamma(x)]$. Let $W = \bigcup \{S_{\lambda_x/2}[x] : x \in F\}$ and let $\{V_i : i \in I_0\}$ be a locally finite open refinement of the cover $\{S_{\lambda_x/2}[x] : x \in F\}$ of W. Define $I \subset I_0$ as follows:

$$I = \{i : i \in I_0 \text{ and } V_i \cap F \neq \varnothing\}.$$

Set $V = \bigcup \{V_i : i \in I\}$: the collection $\{V_i : i \in I\}$ is a locally finite open cover of V. By Lemma 5 there are open sets G_1 and G_2 such that (i) $G_1 \cup G_2 = X$, (ii) $G_1 \cap G_2 = \varnothing$, and (iii) $F \subset G_1 \subset V$. For each $i \in I$ set $V_i^* = G_1 \cap V_i$. Notice that for each index i the open set V_i^* contains infinitely many limit points of X. Hence, reasoning as in Proposition 1 of [5], there is an injection $i \to a_i$ defined on I such that for each i both $a_i \in V_i^*$ and a_i is a limit point of X. We now proceed as in the proof of the main theorem of [5]. There is a collection of pairwise disjoint closed balls $\{K_i : i \in I\}$ such that for each index i, $a_i \in \operatorname{int} K_i \subset V_i^*$. For each i the set

$$U_i = V_i^* - \bigcup_{\substack{j \in I, \\ j \neq i}} K_j$$

is open and contains K_i. Furthermore, $\{U_i : i \in I\}$ is a locally finite refinement of the cover $\{V_i^* : i \in I\}$ of G_1. Since $\{U_i : i \in I\}$ is a refinement of $\{S_{\lambda_x/2}[x] : x \in F\}$, for each i we can find b_i in F such that for each z in U_i, $d(z, b_i) < \frac{1}{2}\lambda_{b_i}$. Again by Lemma 3 for each index i there is $f_i : U_i \to \Gamma(b_i)$ such that (i) $\delta[f_i(K_i), \Gamma(b_i)] \leqslant \varepsilon/2$, (ii) $D(f_i) = \{a_i\}$, and (iii) the limit set multifunction for f_i is both convex valued and u.s.c. If the values of Γ are totally bounded, then each such f_i can be chosen continuous. Let $\{p_i(\cdot) : i \in I\}$ be a partition of unity subordinated to $\{U_i : i \in I\}$. Since whenever $i \neq k$ we have $K_i \cap U_k = \varnothing$, Lemmas 1 and 2 and Cellina's lemma all apply: the function $g_1 : G_1 \to Y$ given by $g_1 = \sum p_i f_i$ satisfies $\delta[g_1, \Gamma | G_1] \leqslant \varepsilon$, and the limit set multifunction for g_1 is both u.s.c. and convex valued. Since $D(g_1) = \{a_i : i \in I\}$, a closed discrete set, g_1 is of Baire class one. As usual, if for each index i the function f_i is continuous, then g_1 is continuous.

Finally, by the first part of the proof there exists $g_2 : G_2 \to Y$ such that (i) $\delta[g_2, \Gamma | G_2] \leqslant \varepsilon$, (ii) $D(g_2)$ is a closed discrete set, and (iii) the limit set multifunction for g_2 is u.s.c. and convex valued. If Γ has totally bounded values, then g_2 can be chosen continuous. The function $f: X \to Y$ defined by

$$\begin{aligned} f(x) &= g_1(x), \quad &\text{if} \quad x \in G_1, \\ &= g_2(x), \quad &\text{if} \quad x \in G_2, \end{aligned}$$

is the desired function.

THEOREM 2. *Let X be a metric space and let $\Gamma: X \to \mathrm{CL}(R^n)$ be a convex valued* u.s.c. *multifunction mapping isolated points to singletons. Then there exists a function $f: X \to R^n$ with a closed graph such that $\delta[f, \Gamma] \leqslant \varepsilon$.*

Proof. Recall that the functions with closed graphs are precisely those with singleton limit sets. Lemma 1 thus applies to this class of functions. Hence, the proof of Theorem 1 goes through intact, except that we invoke Lemma 4 in lieu of Lemma 3.

It should be noticed that in the statement of Theorem 2 it is not claimed that the limit set multifunction for f is u.s.c., nor is it claimed that f is of Baire class one. Indeed, the upper semicontinuity of the limit set multifunction for a function with closed graph implies continuity of the function. On the other hand, by virtue of our next result, the statement that f is of Baire class one is redundant.

THEOREM 3. *Let X be a metric space and let Y be a separable locally compact metric space. If $f: X \to Y$ has a closed graph, then f is of Baire class one.*

Proof. The proof of Theorem 1.6.2. of [10] shows that $D(f)$ is a closed set. Let G be an open subset of Y. Since $D(f)$ is closed, $f^{-1}(G) \cap [X - D(f)]$ is an open subset of X. Since open sets in a metric space are F_σ sets, it remains to show that $f^{-1}(G) \cap D(f)$ is an F_σ set. By our assumptions for Y, G may be represented as a countable union of compact sets $G = \bigcup_{i=1}^{\infty} K_i$. We claim that for each index i the set $f^{-1}(K_i) \cap D(f)$ is a closed subset of X. Suppose $\{x_k\}$ is a sequence in $f^{-1}(K_i) \cap D(f)$ convergent to some point x. Since $D(f)$ is closed, $x \in D(f)$. Since K_i is compact, $\{f(x_k)\}$ has a subsequence convergent to some point y in K_i. By definition $y \in L(f, x)$, and since $L(f, x) = \{f(x)\}$, we have $x \in f^{-1}(K_i)$.

We leave it to the reader to show that Theorem 3 fails if either "locally compact" is replaced by "complete," or separability is not assumed. We close with a most unfortunate fact of life: even for $X = [0, 1]$, if $\dim(Y) > 1$ and $\Gamma: X \to \mathrm{CL}(Y)$ is an u.s.c. compact valued multifunction that is the δ-limit of a sequence of continuous functions, we cannot conclude that Γ has convex values. (If $\dim(Y) = 1$ we can draw this conclusion, provided X is locally connected [1].)

EXAMPLE 3. We present a sequence $\{f_n\}$ of continuous functions on $[0, 1]$ whose graphs converge in the Hausdorff metric to the graph of a compact valued u.s.c. multifunction Γ: $[0, 1] \to \mathrm{CL}(R^2)$ that fails to have convex values. For each $n \in Z^+$ define $f_n: [0, 1] \to R^2$ by

$$f_n(x) = \left(\sin \frac{1}{x}, 1\right), \qquad \text{if} \quad \frac{1}{n\pi} \leqslant x \leqslant 1,$$

$$= (0, n\pi x), \qquad \text{if} \quad 0 \leqslant x < \frac{1}{n\pi}.$$

Let $T = \{(y, 1): -1 \leqslant y \leqslant 1\} \cup \{(0, z): 0 \leqslant z \leqslant 1\}$. Then if $\Gamma: [0, 1] \to \mathrm{CL}(R^2)$ is defined by

$$\Gamma(x) = T, \qquad \text{if} \quad x = 0,$$

$$= \left\{\left(\sin \frac{1}{x}, 1\right)\right\}, \qquad \text{if} \quad 0 < x \leqslant 1,$$

we have $\lim_{n \to \infty} \delta[f_n, \Gamma] = 0$.

4. TWO APPLICATIONS OF THEOREM 1

Let M be a nonempty closed convex subset of a finite dimensional subspace of a normed linear space X. For each x in X the set $P_M(x)$ defined by

$$P_M(x) = \{y: y \in M \text{ and } \|x - y\| = \inf_{m \in M} \|x - m\|\}$$

is a nonempty compact convex subset of M. Moreover, the assignment $x \to P_M(x)$, called the *metric projection* of X onto M [7], is u.s.c. Thus, Theorem 1 says that the restriction of this multifunction to any perfect subset of X admits for each $\varepsilon > 0$ a continuous ε-approximate selection. In particular, if S is a closed star-shaped set in the space whose convex kernel lies in some finite dimensional subspace, then the metric projection of S onto its kernel can be approximated by continuous functions in the Hausdorff metric.

For an application of the general version of Theorem 1, let X be a separable Hilbert space and let C be a closed convex subset of X. For each x in C the *normal cone* to C at x [16] is defined by

$$N_C(x) = \{y: y \in X \text{ and for each } w \in C, y \cdot (w - x) \leqslant 0\}.$$

The normal cone to C at x is a closed convex set closed under addition and multiplication by nonnegative scalars; evidently, the nonzero vectors in $N_C(x)$ determine the closed support hyperplanes to C at x. Although $x \to N_C(x)$ (as a multifunction on C) has a closed graph, it is not in general u.s.c. However, if X is finite dimensional, it is not hard to show using the compactness of the unit ball that the normal cone multifunction is actually u.s.c. Hence, the general version of Theorem 1 ensures for each positive ε the existence of an u.s.c. convex valued multifunction that admits a Baire class one dense selection which ε-approximates $x \to N_C(x)$ on C.

References

1. G. Beer, The approximation of real functions in the Hausdorff metric, to appear.
2. G. Beer, Dense selections, *J. Math. Anal. Appl.* **95** (1983), 416–427.
3. C. Berge, "Topological Spaces," Oliver & Boyd, Edinburgh, 1963.
4. A. Cellina, A theorem on the approximation of compact multivalued mappings, *Rend. Accad. Naz. Lincei* (8) **47** (1969), 429–433.
5. A. Cellina, A further result on the approximation of set valued mappings, *Rend. Accad. Naz. Lincei* (8) **48** (1970), 412–416.
6. A. Cellina and A. Lasota, A new approach to the definition of topological degree for multivalued mappings, *Rend. Accad. Naz. Lincei* (8) **47** (1969), 434–440.
7. F. Deutsch and P. Kenderov, Continuous selections and approximate selections for set valued mappings and applications to metric projections, *SIAM J. Math. Anal.*, in press.
8. S. Dolecki and S. Rolewicz, Metric characterizations of upper semicontinuity, *J. Math. Anal. Appl.* **69** (1979), 146–152.
9. J. Dugundji, "Topology," Allyn & Bacon, Boston, 1966.
10. T. Hamlett and L. Herrington, "The Closed Graph and p-Closed Graph Properties in General Topology," A.M.S. Contemporary Math. Series, Vol. 3, Amer. Math. Soc., Providence, R.I., 1981.
11. K. Kuratowski, "Topology," Academic Press, New York, 1966.
12. E. Michael, Selected selection theorems, *Amer. Math. Monthly* **63** (1956), 233–237.
13. S. Nadler, "Hyperspaces of Sets," Dekker, New York, 1978.
14. C. Olech, Approximation of set-valued functions by continuous functions, *Colloq. Math.* **19** (1968), 285–293.
15. S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, *J. Math. Anal. Appl.* **62** (1978), 104–113.
16. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N. J., 1970.
17. D. Wagner, Survey of measurable selections, *SIAM J. Control Optim.* **15** (1977), 859–903.
18. D. Wagner, Survey of measurable selections: An update, *in* "Measure Theory Oberwolfach 1979," Springer Verlag, Berlin, 1980.